

GENERALIZED RESOLVENTS OF ISOMETRIC OPERATORS IN PONTRYAGIN SPACES

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ABSTRACT. An isometric operator V in a Pontryagin space \mathcal{H} is called standard, if its domain and the range are nondegenerate subspaces in \mathcal{H} . Generalized resolvents of standard isometric operators were described in [11]. In the present paper generalized resolvents of non-standard Pontryagin space isometric operators are described. The method of the proof is based on the notion of boundary triplet of isometric operators in Pontryagin spaces. In the Hilbert space setting the notion of boundary triplet for isometric operators was introduced in [17].

1. INTRODUCTION

Unitary operators in Pontryagin spaces and the problem of continuation of an isometric operators V were studied in papers [3], [12], [13], [15], [16]. An isometric operator V in a Pontryagin space \mathcal{H} is called standard, if its domain and the range are nondegenerate subspaces in \mathcal{H} . A description of generalized resolvents of a standard isometric operator was given in [11]. For a nonstandard isometric operator, the approach presented in [19], [20] leads to significant technical problems related to the necessity to consider unitary linear relations in a Pontryagin space.

We propose another approach to the theory of extensions of isometric operators in Pontryagin spaces that is based on the notion of the boundary triplet of an isometric operator. In the case of a Hilbert space \mathcal{H} , this notion was introduced and applied to the classical problems of analysis in works by M. M. Malamud and V. I. Mogilevskii [17] and [18]. For a Pontryagin space setting the definition of boundary triplet of isometric operator given in [4] is a partial case of the definition of boundary relation in [10]. Boundary triplets considered in [5] had the property that the auxiliary spaces of that triplets were Hilbert spaces. Such objects were sufficient in order to give a description of generalized resolvents of V corresponding to unitary extensions of V acting in wider spaces $\tilde{\mathcal{H}}$ with the same negative index as \mathcal{H} .

In the present paper we give a description of generalized resolvents of an isometric operator V which correspond to exit space unitary extensions of V acting in spaces $\tilde{\mathcal{H}}$ with negative index $\tilde{\kappa}$ exceeding the negative index of \mathcal{H} . It turned out that the notion of boundary triplet with Hilbert auxiliary space introduced in [5] is not sufficient for this purpose. That is why we extend the notion of boundary triplet for Pontryagin space isometric operator to the case where the auxiliary space is a Pontryagin space.

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We introduce the notion of the Weyl function of an isometric operator, which generalizes the appropriate definition from [17] and study its properties. This allows us to describe the properties of extensions of the operator V , as well as generalized resolvents of the isometric operator V in a Pontryagin space.

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2. PRELIMINARY INFORMATION

2.1. Linear relations. We recall some information about linear relations from [6], [10]. Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces. A linear relation (l.r.) T from \mathcal{H}_1 to \mathcal{H}_2 is a linear subspace in $\mathcal{H}_1 \times \mathcal{H}_2$. If the linear operator T is identified with its graph, then the set $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of linear bounded operators from \mathcal{H}_1 to \mathcal{H}_2 is contained in the set of linear relations from \mathcal{H}_1 to \mathcal{H}_2 . In what follows, we interpret the linear relation $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ as a multivalued linear mapping from \mathcal{H}_1 to \mathcal{H}_2 . If $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ we say that T is a linear relation in \mathcal{H} .

For the linear relation $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, we denote by $\text{dom } T$, $\ker T$, $\text{ran } T$, and $\text{mul } T$ the domain, the kernel, the range, and the multivalued part of T , respectively. The inverse relation T^{-1} is a linear relation from \mathcal{H}_2 to \mathcal{H}_1 defined by the equality

$$T^{-1} = \left\{ \begin{bmatrix} f' \\ f \end{bmatrix} : \begin{bmatrix} f \\ f' \end{bmatrix} \in T \right\}.$$

The sum $T + S$ of two linear relations T and S is defined by

$$(2.1) \quad T + S = \left\{ \begin{bmatrix} f \\ g + h \end{bmatrix} : \begin{bmatrix} f \\ g \end{bmatrix} \in T, \begin{bmatrix} f \\ h \end{bmatrix} \in S \right\}.$$

Let \mathcal{H}_1 and \mathcal{H}_2 be Banach spaces. By $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we denote the set of all linear bounded operators from \mathcal{H}_1 to \mathcal{H}_2 ; $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. We recall that the point $\lambda \in \mathbb{C}$ is called a point of regular type of an operator $T \in \mathcal{B}(\mathcal{H})$, if there exists $c_\lambda > 0$ such that

$$(2.2) \quad \|(T - \lambda I)f\|_{\mathcal{H}} \geq c_\lambda \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

If $\text{ran}(T - \lambda I) = \mathcal{H}$ and (2.2) holds, then λ is called a regular point of the operator T . Let $\rho(T)$ ($\widehat{\rho}(T)$) be the set of regular (regular type) points of the operator T and let

$$(2.3) \quad R_\lambda(T) := (T - \lambda I)^{-1}, \quad \lambda \in \rho(T).$$

2.2. Linear relations in Pontryagin spaces. Let \mathcal{H} be a Hilbert space, and let $j_{\mathcal{H}}$ be a signature operator in \mathcal{H} , i.e., $j_{\mathcal{H}} = j_{\mathcal{H}}^* = j_{\mathcal{H}}^{-1}$. We interpret the space \mathcal{H} as a Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ (see [3]), in which the indefinite scalar product is defined by the equality $[\varphi, \psi]_{\mathcal{H}} = (j_{\mathcal{H}}\varphi, \psi)_{\mathcal{H}}$. The signature operator $j_{\mathcal{H}}$ can be presented in the form $j_{\mathcal{H}} = P_+ - P_-$, where P_+ and P_- orthoprojectors in \mathcal{H} . In the case where P_- is finite-dimensional, and $\dim P_- \mathcal{H} = \kappa$, the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ is called a Pontryagin space with negative index κ , which is denoted by $\text{ind}_- \mathcal{H} = \kappa$.

Consider two Pontryagin spaces $(\mathcal{H}_1, j_{\mathcal{H}_1})$ and $(\mathcal{H}_2, j_{\mathcal{H}_2})$ and a linear relation T from \mathcal{H}_1 to \mathcal{H}_2 . Then the adjoint linear relation $T^{[*]}$ consists of pairs $\begin{bmatrix} g_2 \\ g_1 \end{bmatrix} \in \mathcal{H}_2 \times \mathcal{H}_1$

such that

$$[f_2, g_2]_{\mathcal{H}_2} = [f_1, g_1]_{\mathcal{H}_1}, \text{ for all } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in T.$$

If T^* is the l.r. adjoint to T considered as a l.r. from the Hilbert space \mathcal{H}_1 to the Hilbert space \mathcal{H}_2 , then $T^{[*]} = j_{\mathcal{H}_1} T^* j_{\mathcal{H}_2}$.

The l.r. $T^{[*]}$ satisfies the equalities

$$(2.4) \quad (\text{dom } T)^{[\perp]} = \text{mul } T^{[*]}, \quad (\text{ran } T)^{[\perp]} = \ker T^{[*]},$$

where the sign $[\perp]$ means the orthogonality in a Pontryagin space.

Definition 2.1. A linear relation T from a Pontryagin space $(\mathcal{H}_1, j_{\mathcal{H}_1})$ to a Pontryagin space $(\mathcal{H}_2, j_{\mathcal{H}_2})$ is called isometric, if, for all $\begin{bmatrix} \varphi \\ \varphi' \end{bmatrix} \in T$, the equality

$$(2.5) \quad [\varphi', \varphi']_{\mathcal{H}_2} = [\varphi, \varphi]_{\mathcal{H}_1}$$

holds. It is called a contractive one, if the equality (2.5) is replaced by an inequality with the sign \leq . A linear relation T from $(\mathcal{H}_1, j_{\mathcal{H}_1})$ to $(\mathcal{H}_2, j_{\mathcal{H}_2})$ is called unitary, if $T^{-1} = T^{[*]}$. It follows from (2.5) that a linear relation T is isometric iff $T^{-1} \subset T^{[*]}$.

As is known [3], the sets $\mathbb{D} \setminus \widehat{\rho}(T)$ and $\mathbb{D}_e \setminus \widehat{\rho}(T)$ for an isometric operator T in a Pontryagin space with $\text{ind } _ \mathcal{H} = \kappa$ consist of at most κ points, which belong to $\sigma_p(T)$.

The definition of unitary relation was first given in [21], where the following assertion was proved.

Proposition 2.2. *If T is a unitary relation, then*

- (1) *$\text{dom } T$ is closed iff $\text{ran } T$ is closed;*
- (2) *the equalities $\ker T = \text{dom } T^{[\perp]}$, $\text{mul } T = \text{ran } T^{[\perp]}$ hold.*

Proposition 2.2 yields the following result.

Corollary 2.3. *If T is a unitary relation in a Pontryagin space, then $\text{mul } T \neq \{0\}$ if and only if $\ker T \neq \{0\}$. In this case, $\dim \text{mul } T = \dim \ker T$.*

3. BOUNDARY TRIPLETS FOR AN ISOMETRIC OPERATOR IN A PONTYAGIN SPACE

3.1. Boundary triplets and description of extensions of an isometric operator in a Pontryagin space. In the case where \mathcal{H} is a Hilbert space, the definition of the boundary triplet for an isometric operator was introduced in [17]. We note that the notion of the boundary triplet of an isometric operator, which will be introduced below in Definition 3.1, is a partial case of the notion of the boundary relation of an isometric operator in a Pontryagin space [4]. The notion of boundary triplets for symmetric operator was introduced in [8] (see also [14] and references therein).

Definition 3.1. Let $\mathcal{H}, \mathfrak{N}_1$ and \mathfrak{N}_2 be Pontryagin spaces with negative indices κ and $\text{ind } _ \mathfrak{N}_1 = \text{ind } _ \mathfrak{N}_2 = \kappa_1$ respectively. Let an operator $V : \mathcal{H} \rightarrow \mathcal{H}$ be an isometry in \mathcal{H} . The collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ is called the boundary triplet of the isometric operator V , if

- 1) the following Green's generalized identity holds:

$$(3.1) \quad [f', g']_{\mathcal{H}} - [f, g]_{\mathcal{H}} = [\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g}]_{\mathfrak{N}_1} - [\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g}]_{\mathfrak{N}_2},$$

$$\text{where } \widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix}, \widehat{g} = \begin{bmatrix} g \\ g' \end{bmatrix} \in V^{-[*]};$$

2) the mapping $\Gamma = (\Gamma_1, \Gamma_2)^T : V^{-[*]} \rightarrow \mathfrak{N}_1 \oplus \mathfrak{N}_2$ is surjective.

For an isometric operator, it is convenient to define the defect subspace $\mathfrak{N}_\lambda(V)$ as follows:

$$(3.2) \quad \mathfrak{N}_\lambda(V) := \ker \left(I - \lambda V^{[*]} \right) = \left\{ f_\lambda : \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} \in V^{-[*]} \right\}.$$

For $\lambda \in \widehat{\rho}(V)$, $\mathfrak{N}_\lambda(V)$ is a closed subspace in \mathcal{H} [3].

We also set

$$(3.3) \quad \widehat{\mathfrak{N}}_\lambda(V) := \left\{ \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} : f_\lambda \in \mathfrak{N}_\lambda(V) \right\}.$$

It follows from (3.2) that $\widehat{\mathfrak{N}}_\lambda(V) \subset V^{-[*]}$.

Let θ be a linear relation from \mathfrak{N}_2 to \mathfrak{N}_1 . We define the extension V_θ of the operator V by the equality

$$(3.4) \quad V_\theta = \left\{ \widehat{f} \in V^{-[*]} : \begin{bmatrix} \Gamma_2 \widehat{f} \\ \Gamma_1 \widehat{f} \end{bmatrix} \in \theta \right\}.$$

The extension V_θ is, generally speaking, a linear relation from \mathcal{H} to \mathcal{H} .

We define two extension V_1 and V_2 of the operator V :

$$(3.5) \quad V_i = \left\{ \widehat{f} \in V^{-[*]} : \Gamma_i \widehat{f} = 0 \right\}, \quad i = 1, 2.$$

We note also that

$$(3.6) \quad V = \left\{ \widehat{f} \in V^{-[*]} : \Gamma_1 \widehat{f} = 0 \text{ and } \Gamma_2 \widehat{f} = 0 \right\}.$$

(A) Here and further on the sets

$$(3.7) \quad \Lambda_1 = \{ \lambda \in \mathbb{D}_e : \widehat{\mathfrak{N}}_\lambda(V) \cap V_1 \neq \{0\} \} = \sigma(V_1) \cap \mathbb{D}_e;$$

$$(3.8) \quad \Lambda_2 = \{ \lambda \in \mathbb{D} : \widehat{\mathfrak{N}}_\lambda(V) \cap V_2 \neq \{0\} \} = \sigma(V_2) \cap \mathbb{D}.$$

will be supposed to be discrete, isolated. Denote $\mathcal{D}_1 := \mathbb{D}_e \setminus \Lambda_1$ and $\mathcal{D}_2 := \mathbb{D} \setminus \Lambda_2$ the subsets of regular points of these extensions.

Lemma 3.2. *Let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet of the isometric operator V . Then:*

(1) For all $\lambda \in \mathcal{D}_1 = \rho(V_1) \cap \mathbb{D}_e$

$$(3.9) \quad V^{-[*]} = V_1 \dot{+} \widehat{\mathfrak{N}}_\lambda(V);$$

(2) For all $\lambda \in \mathcal{D}_2 = \rho(V_2) \cap \mathbb{D}$

$$(3.10) \quad V^{-[*]} = V_2 \dot{+} \widehat{\mathfrak{N}}_\lambda(V).$$

Proof. We prove equality (3.9) (equality (3.10) can be proved analogously). For this purpose, we set the inclusion $V^{-[*]} \subset V_1 \dot{+} \widehat{\mathfrak{N}}_\lambda(V)$. Consider a pair of vectors

$\begin{bmatrix} f \\ f' \end{bmatrix} \in V^{-[*]}$. Let $f_1 = (V_1 - \lambda)^{-1}(f' - \lambda f)$ be a solution of the equation

$$f' - \lambda f = f'_1 - \lambda f_1, \text{ where } \begin{bmatrix} f'_1 \\ f_1 \end{bmatrix} \in V_1,$$

which is determined uniquely for $\lambda \in \mathcal{D}_1$. Then $f' - f'_1 = \lambda(f - f_1)$, i.e., $\begin{bmatrix} f - f_1 \\ \lambda(f - f_1) \end{bmatrix} \in V^{-[*]}$ and, hence, $f - f_1 \in \mathfrak{N}_\lambda(V)$. Since the inverse inclusion is obvious, equality (3.9) is proved. \square

The following theorem gives a description of proper extensions of the operator V , i.e., such that $V \subset V_\theta \subset V^{-[*]}$.

Theorem 3.3. *Let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet for V , let θ be a linear relation from \mathfrak{N}_2 to \mathfrak{N}_1 , and let V_θ be the corresponding extension of the operator V . Then:*

- (1) *the inclusion $V_{\theta_1} \subset V_{\theta_2}$ is equivalent to the inclusion $\theta_1 \subset \theta_2$;*
- (2) *$V_{\theta^{-[*]}} = V_\theta^{-[*]}$;*
- (3) *V_θ is a unitary extension of the operator V , iff θ is the graph of a unitary l.r. from \mathfrak{N}_2 to \mathfrak{N}_1 ;*
- (4) *V_θ is an isometric extension of the operator V , iff θ is the graph of an isometric l.r. from \mathfrak{N}_2 to \mathfrak{N}_1 ;*
- (5) *V_θ is a coisometric extension of the operator V , iff θ is the graph of a coisometric l.r. from \mathfrak{N}_2 to \mathfrak{N}_1 ;*
- (6) *V_θ is a contraction, iff θ is a contraction;*
- (7) *V_θ is an expansion, iff θ is an expansion.*

Proof. Assertion 1) follows obviously from the definition of V_{θ_1} and V_{θ_2} .

2) We take $\begin{bmatrix} f \\ f' \end{bmatrix} \in V_\theta$ and $\begin{bmatrix} g \\ g' \end{bmatrix} \in V_\theta^{-[*]}$. Then $\begin{bmatrix} g' \\ g \end{bmatrix} \in V_\theta^{[*]}$. From (3.1), we obtain

$$0 = [f', g'] - [f, g] = \left[\Gamma_1 \hat{f}, \Gamma_1 \hat{g} \right]_{\mathfrak{N}_1} - \left[\Gamma_2 \hat{f}, \Gamma_2 \hat{g} \right]_{\mathfrak{N}_2}$$

Since $\begin{bmatrix} \Gamma_2 \hat{f} \\ \Gamma_1 \hat{f} \end{bmatrix} \in \theta$, we have $\begin{bmatrix} \Gamma_1 \hat{g} \\ \Gamma_2 \hat{g} \end{bmatrix} \in \theta^{[*]}$ or $\begin{bmatrix} \Gamma_2 \hat{g} \\ \Gamma_1 \hat{g} \end{bmatrix} \in \theta^{-[*]}$, which means $\hat{g} \in V_{\theta^{-[*]}}$.

Hence, we show that $V_\theta^{-[*]} \subset V_{\theta^{-[*]}}$.

The inverse assertion can be proved by inversion of the above reasoning.

3) Let $V_\theta^{-[*]} = V_\theta$, i.e., let V_θ be a unitary extension of the operator V . Using the first assertion of this Theorem, we obtain $\theta^{-[*]} = \theta$. Conversely, we set $\theta^{-[*]} = \theta$ and, by the first assertion of this Theorem, arrive at $V_\theta^{-[*]} = V_\theta$.

4) and 5) are proved analogously. Assume that V_θ is a coisometry, i.e., $V_\theta^{-1} \supset V_\theta^{[*]}$. Then, by virtue of item 2), $V_{\theta^{-[*]}} = V_\theta^{-[*]} \subset V_\theta$. By virtue of assertion 1), we obtain $\theta^{-[*]} \subset \theta$, i.e., θ is a coisometry.

6) Let V_θ be a contraction. Then, for $\hat{f} = \begin{bmatrix} f \\ f' \end{bmatrix} \in V_\theta$, formula (3.1) yields

$$0 \geq [f', f'] - [f, f] = \left[\Gamma_1 \hat{f}, \Gamma_1 \hat{f} \right]_{\mathfrak{N}_1} - \left[\Gamma_2 \hat{f}, \Gamma_2 \hat{f} \right]_{\mathfrak{N}_2}.$$

We obtain $\left[\Gamma_1 \hat{f}, \Gamma_1 \hat{g} \right]_{\mathfrak{N}_1} \leq \left[\Gamma_2 \hat{f}, \Gamma_2 \hat{g} \right]_{\mathfrak{N}_2}$. This means that θ is a contraction.

7) is proved analogously 6). \square

Corollary 3.4. *The extensions V_1 and V_2 of the operator V are connected by the formula*

$$(3.11) \quad V_2 = V_1^{-[*]}.$$

Proof. It is gotten by using the second assertion of the previous Theorem with $\theta = 0$. Because in this case $V_\theta = V_1$ and $V_{\theta-[*]} = V_2$. \square

3.2. γ -field and Weyl function. The notion of the Weyl function of an isometric operator V in a Hilbert space, which allows one to describe the analytic properties of extensions of the operator V , was introduced in [17]. In this section, we will generalize this notion to the case of isometric operator V in a Pontryagin space.

Lemma 3.5. *Let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet for V , and let V_1 and V_2 be the extensions of the isometric operator V that are defined in (3.5). Then the mappings $\Gamma_j \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) : \widehat{\mathfrak{N}}_\lambda(V) \rightarrow \mathfrak{N}_j$ $j = 1, 2$, are bounded and boundedly invertible for $\lambda \in \mathcal{D}_j$.*

In this case, the operator-functions

$$(3.12) \quad \gamma_j(\lambda) := \pi_1 \widehat{\gamma}_j(\lambda) = \pi_1 \left(\Gamma_j \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) \right)^{-1}$$

satisfy the equality

$$(3.13) \quad \gamma_j(\lambda) = (I + (\lambda - \mu)(V_j - \lambda)^{-1})\gamma_j(\mu), \quad \text{for } \lambda, \mu \in \mathcal{D}_j, \text{ and } j = 1, 2.$$

The operator-functions $\gamma_j(\cdot)$ are called γ -fields for the l.r. $V^{-[]}$.*

Proof. First, we will show that the mapping $\Gamma : V^{-[*]} \rightarrow \begin{bmatrix} \mathfrak{N}_1 \\ \mathfrak{N}_2 \end{bmatrix}$ is closed. Let

$\widehat{f} = \begin{bmatrix} f_n \\ f'_n \end{bmatrix} \in V^{-[*]}$ and $\widehat{f}_n \rightarrow 0$. Then $\Gamma \widehat{f}_n = \begin{bmatrix} \Gamma_1 \widehat{f}_n \\ \Gamma_2 \widehat{f}_n \end{bmatrix} \rightarrow \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =: h$. From (3.1), we obtain $[h_1, \Gamma_1 \widehat{g}]_{\mathfrak{N}_1} - [h_2, \Gamma_2 \widehat{g}]_{\mathfrak{N}_2} = 0$. The surjectivity of Γ and nondegeneracy of \mathfrak{N}_1 and \mathfrak{N}_2 , imply that $h = 0$.

Since $\text{dom } \Gamma = V^{-[*]}$, the operator Γ is bounded by the closed graph Banach theorem. Hence, Γ_1 and Γ_2 are bounded as well.

By virtue of equality (3.9) and the surjectivity of Γ , the mapping

$$\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) : \widehat{\mathfrak{N}}_\lambda(V) \rightarrow \mathfrak{N}_1$$

acts on all \mathfrak{N}_1 . Hence, $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(V)$ is boundedly invertible.

Analogously, we can prove the bounded invertibility of $\Gamma_2 \upharpoonright \widehat{\mathfrak{N}}_\lambda(V)$. Hence, $\gamma_j(\lambda)$ for $\lambda \in \mathcal{D}_j$ $j = 1, 2$ are defined properly.

We now prove identity (3.13). For definiteness, we take $j = 1$ and will prove that

$$\gamma_1(\lambda) = (I + (\lambda - \mu)(V_1 - \lambda)^{-1})\gamma_1(\mu), \quad \text{for } \lambda, \mu \in \mathcal{D}_1.$$

Consider the vector $g_\mu = \gamma_1(\mu)h_1 \in \mathfrak{N}_\mu(V)$, where $h_1 \in \mathfrak{N}_1$. Then there exists the vector $h_2 \in \mathfrak{N}_2$ such that $\Gamma \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$. We set

$$f_\lambda = g_\mu + (\lambda - \mu)(V_1 - \lambda)^{-1}g_\mu.$$

Then

$$(3.14) \quad \widehat{f}_\lambda = \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} + (\lambda - \mu) \begin{bmatrix} (V_1 - \lambda)^{-1}g_\mu \\ (I + \lambda(V_1 - \lambda)^{-1})g_\mu \end{bmatrix}.$$

In this equality,

$$\widehat{g}_\mu = \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} \in \widehat{\mathfrak{N}}_\mu(V) \subset V^{-[*]}, \begin{bmatrix} (V_1 - \lambda)^{-1}g_\mu \\ (I + \lambda(V_1 - \lambda)^{-1})g_\mu \end{bmatrix} \in V_1 \subset V^{-[*]}.$$

Thus, $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(V)$.

Below, we will use an equality that follows from (3.5)

$$(3.15) \quad \Gamma_1 \left[\begin{pmatrix} (V_1 - \lambda)^{-1} g_\mu \\ (I + \lambda(V_1 - \lambda)^{-1}) g_\mu \end{pmatrix} \right] = 0$$

Equalities (3.14) and (3.15) yield

$$\Gamma_1 \widehat{f}_\lambda = \Gamma_1 \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} + (\lambda - \mu) \Gamma_1 \begin{bmatrix} (V_1 - \lambda)^{-1} g_\mu \\ (I + \lambda(V_1 - \lambda)^{-1}) g_\mu \end{bmatrix} = h_1$$

Hence, $f_\lambda = \pi_1 \Gamma_1^{-1} h_1 = \gamma_1(\lambda) h_1$. This proves (3.13). \square

The previous lemma implies that it is possible to define the operator-functions $M_1(\cdot)$ and $M_2(\cdot)$:

$$(3.16) \quad M_1(\lambda) \Gamma_1 \upharpoonright \mathfrak{N}_\lambda(V) = \Gamma_2 \upharpoonright \mathfrak{N}_\lambda(V), \quad \lambda \in \mathcal{D}_1;$$

$$(3.17) \quad M_2(\lambda) \Gamma_2 \upharpoonright \mathfrak{N}_\lambda(V) = \Gamma_1 \upharpoonright \mathfrak{N}_\lambda(V), \quad \lambda \in \mathcal{D}_2.$$

It follows from definition (3.12) of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ that $M_1(\lambda)$ and $M_2(\lambda)$ are defined properly, and

$$(3.18) \quad M_1(\lambda) := \Gamma_2 \widehat{\gamma}_1(\lambda), \quad \lambda \in \mathcal{D}_1;$$

$$(3.19) \quad M_2(\lambda) := \Gamma_1 \widehat{\gamma}_2(\lambda), \quad \lambda \in \mathcal{D}_2.$$

In what follows, we need the Schur class S and the generalized Schur class S_κ of functions. Their definition is given below.

Definition 3.6. A function $s(\cdot)$ defined and holomorphic in a domain $\mathfrak{h}_s \subset \mathbb{D}$ belongs to the class $S_\kappa(\mathfrak{N}_1, \mathfrak{N}_2)$, if the kernel

$$K_\mu(\lambda) = \frac{1 - s(\mu)^* s(\lambda)}{1 - \lambda \bar{\mu}}$$

has κ negative squares, i.e. for all $\lambda_1, \dots, \lambda_n \in \mathfrak{h}_s$ and $u_1, \dots, u_n \in \mathfrak{N}_1$ the matrix $([K_{\lambda_j}(\lambda_i) u_i, u_j]_{\mathfrak{N}_1})_{i,j=1}^n$ has at most κ negative eigenvalues. For at least one such choice, it has exactly κ negative eigenvalues.

In particular, if \mathfrak{N}_1 and \mathfrak{N}_2 are Hilbert spaces then the $[\mathfrak{N}_1, \mathfrak{N}_2]$ -valued function $s(\cdot)$ belongs to the class $S(\mathfrak{N}_1, \mathfrak{N}_2)$, if the kernel $K_\mu(\lambda)$ is positive definite everywhere in \mathbb{D} . As is known, the last condition is equivalent to that $s(\cdot)$ is holomorphic in \mathbb{D} , and $\|s(\lambda)\| \leq 1$ for all $\lambda \in \mathbb{D}$.

Proposition 3.7. Let $M_2(\cdot)$ be the operator-function defined by (3.19). Then $M_2(\cdot) \in S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$.

Proof. Let λ_j be some points from \mathcal{D}_2 , $j = 1, \dots, n$. We denote $h_j := \Gamma_2 \widehat{f}_{\lambda_j}$. Then $\Gamma_1 \widehat{f}_{\lambda_j} = M_2(\lambda_j) h_j$. From (3.1) for $\widehat{f}_{\lambda_j} \in \mathfrak{N}_{\lambda_j}(V)$ and $\widehat{f}_{\lambda_k} \in \mathfrak{N}_{\lambda_k}(V)$, we have

$$(\lambda_j \bar{\lambda}_k - 1) [f_{\lambda_j}, f_{\lambda_k}] = [M_2(\lambda_j) h_j, M_2(\lambda_k) h_k]_{\mathfrak{N}_1} - [h_j, h_k]_{\mathfrak{N}_2}.$$

Now we construct the quadratic form

$$\sum_{j,k=1}^n \left[\frac{I - M_2(\lambda_k)^{[*]} M_2(\lambda_j)}{1 - \lambda_j \bar{\lambda}_k} h_j, h_k \right]_{\mathfrak{N}_2} \xi_j \bar{\xi}_k = \sum_{j,k=1}^n [f_{\lambda_j}, f_{\lambda_k}]_{\mathfrak{H}} \xi_j \bar{\xi}_k.$$

Since \mathcal{H} has the negative index κ , and since the reduced quadratic form has at most κ negative squares and exactly κ negative squares for some collection f_{λ_j} , we have $M_2(\cdot) \in S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$. \square

Proposition 3.8. *The following relations hold:*

$$(3.20) \quad -\frac{I - M_1(\mu)^{[*]}M_1(\lambda)}{1 - \lambda\bar{\mu}} = \gamma_1(\mu)^{[*]}\gamma_1(\lambda), \quad \lambda, \mu \in \mathcal{D}_1;$$

$$(3.21) \quad \frac{I - M_2(\mu)^{[*]}M_2(\lambda)}{1 - \lambda\bar{\mu}} = \gamma_2(\mu)^{[*]}\gamma_2(\lambda), \quad \lambda, \mu \in \mathcal{D}_2;$$

$$(3.22) \quad \frac{M_1(\mu)^{[*]} - M_2(\lambda)}{1 - \lambda\bar{\mu}} = \gamma_1(\mu)^{[*]}\gamma_2(\lambda), \quad \lambda \in \mathcal{D}_2, \mu \in \mathcal{D}_1;$$

$$(3.23) \quad \frac{M_1(\lambda) - M_2(\mu)^{[*]}}{1 - \lambda\bar{\mu}} = \gamma_2(\mu)^{[*]}\gamma_1(\lambda), \quad \lambda \in \mathcal{D}_1, \mu \in \mathcal{D}_2.$$

Proof. We now prove 3.20 and 3.22, because 3.21 is proved analogously to 3.20, and 3.23 is a consequence of 3.22.

Let $\lambda, \mu \in \mathcal{D}_1$ and $h_1, h'_1 \in \mathfrak{N}_1$. Then formula (3.18) yields

$$\Gamma \begin{bmatrix} \gamma_1(\lambda)h_1 \\ \lambda\gamma_1(\lambda)h_1 \end{bmatrix} = \begin{bmatrix} h_1 \\ M_1(\lambda)h_1 \end{bmatrix} \quad \text{and} \quad \Gamma \begin{bmatrix} \gamma_1(\mu)h'_1 \\ \mu\gamma_1(\mu)h'_1 \end{bmatrix} = \begin{bmatrix} h'_1 \\ M_1(\mu)h'_1 \end{bmatrix}.$$

Using this identities and setting (3.1) $\hat{f} = \hat{\gamma}_1(\lambda)h_1$ and $\hat{g} = \hat{\gamma}_1(\mu)h'_1$, we obtain

$$(\lambda\bar{\mu} - 1)[\gamma_1(\lambda)h_1, \gamma_1(\mu)h'_1]_{\mathcal{H}} = [h_1, h'_1]_{\mathfrak{N}_1} - [M_1(\lambda)h_1, M_1(\mu)h'_1]_{\mathfrak{N}_2}$$

or

$$(\lambda\bar{\mu} - 1)[\gamma_1(\mu)^{[*]}\gamma_1(\lambda)h_1, h'_1]_{\mathcal{H}} = [(I - M_1(\mu)^{[*]}M_1(\lambda))h_1, h'_1]_{\mathfrak{N}_1}.$$

From whence, we obtain equality (3.20).

Let now $\lambda \in \mathcal{D}_2$, $\mu \in \mathcal{D}_1$ and let $h_1 \in \mathfrak{N}_1$ and $h_2 \in \mathfrak{N}_2$. Then formulas (3.18) and (3.19) yield

$$\Gamma \begin{bmatrix} \gamma_2(\lambda)h_2 \\ \lambda\gamma_2(\lambda)h_2 \end{bmatrix} = \begin{bmatrix} M_2(\lambda)h_2 \\ h_2 \end{bmatrix} \quad \text{and} \quad \Gamma \begin{bmatrix} \gamma_1(\mu)h_1 \\ \mu\gamma_1(\mu)h_1 \end{bmatrix} = \begin{bmatrix} h_1 \\ M_1(\mu)h_1 \end{bmatrix}.$$

From (3.1), we obtain

$$(\lambda\bar{\mu} - 1)[\gamma_2(\lambda)h_2, \gamma_1(\mu)h_1]_{\mathcal{H}} = [M_2(\lambda)h_2, h_1]_{\mathfrak{N}_1} - [h_2, M_1(\mu)h_1]_{\mathfrak{N}_2}.$$

This yields identity (3.22). \square

Definition 3.9. The isometric operator V in \mathcal{H} is called simple, if

$$\overline{\text{span}}\{\mathfrak{N}_\lambda(V) : \lambda \in \hat{\rho}(V)\} = \mathcal{H}.$$

If the isometric operator V in a Pontryagin space \mathcal{H} is simple, then $\mathbb{D} \cup \mathbb{D}_e \in \hat{\rho}(V)$ (see [3]).

Theorem 3.10. *Let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet of a simple isometric operator V , and let $M_1(\cdot)$ and $M_2(\cdot)$ be the functions defined by equations (3.16) and (3.17). Then the set of poles of the operator-function $M_1(\cdot)$ in \mathbb{D}_e coincide with Λ_1 , and the set of poles of the operator-function $M_2(\cdot)$ in \mathbb{D} coincide with Λ_2 .*

Proof. It follows from (3.20) that if λ_0 is a pole of the operator-function $M_1(\cdot)$, then it is a singular point for $\gamma_1(\cdot)$, i.e. $\lambda_0 \in \Lambda_1$.

Let now $\lambda_0 \in \Lambda_1$. Then

$$(V_1 - \lambda)^{-1} = \frac{A_{-n}}{(\lambda - \lambda_0)^n} + \dots + \frac{A_{-1}}{\lambda - \lambda_0} + \dots$$

Let us assume that $M_1(\lambda)$ is holomorphic at the point λ_0 . Then the equality

$$-\frac{I - M_1(\mu)^{[*]} M_1(\lambda)}{1 - \lambda \bar{\mu}} = \gamma_1(\mu)^{[*]} (I + (\lambda - \mu')(V_1 - \lambda)^{-1}) \gamma_1(\mu'),$$

implies that $[A_{-i} \gamma_1(\mu') h'_1, \gamma_1(\mu) h_1] = 0$ for all $i = 1, \dots, n$, $\mu, \mu' \in \mathcal{D}_1$ and any $h_1, h'_1 \in \mathfrak{N}_1$.

The equality

$$\frac{M_1(\lambda) - M_2(\mu)^{[*]}}{1 - \lambda \bar{\mu}} = \gamma_2(\mu)^{[*]} (I + (\lambda - \mu')(V_1 - \lambda)^{-1}) \gamma_1(\mu'),$$

yields $[A_{-i} \gamma_1(\mu') h_1, \gamma_2(\mu) h_2] = 0$ for all $\mu' \in \mathcal{D}_1$, $\mu \in \mathcal{D}_2$ and any $h_1 \in \mathfrak{N}_1$, $h_2 \in \mathfrak{N}_2$. By virtue of the simplicity of the operator V ,

$$\overline{\text{span}}\{\mathfrak{N}_\lambda(V) : \lambda \in \mathcal{D}_1 \cup \mathcal{D}_2\} = \mathcal{H}.$$

Hence, all $A_{-i} = 0$ for $i = 1, \dots, n$. But this contradicts the assumption that $\lambda_0 \in \Lambda_1$.

The second assertion of this theorem is proved analogously. \square

Let us say

$$(3.24) \quad M_i^\#(\lambda) := M_i(1/\bar{\lambda})^{[*]} \text{ and } \gamma_i^\#(\lambda) := \gamma_i(1/\bar{\lambda})^{[*]},$$

where $i = 1, 2$.

Theorem 3.11. *Under the assumptions of Theorem 3.10, the equality*

$$(3.25) \quad M_1(\lambda) = M_2^\#(\lambda) \text{ for } \lambda \in \mathcal{D}_1$$

holds.

Proof. Let us take $\lambda \in \mathbb{D}_e \setminus (\Lambda_1 \cup \Lambda_2^\#)$, where $\Lambda_2^\#$ is the set symmetric to the set Λ_2 relative to the unite disc. Setting $\mu = 1/\bar{\lambda}$ in (3.23), we obtain (3.25). Equality (3.25) for $\lambda \in \mathcal{D}_1$ can be obtained by the analytic continuation of the function $M_2^\#(\lambda) = M_1(\lambda)$ into the points $\lambda \in \mathcal{D}_1 \cap \Lambda_2^\#$ and by the application of Theorem 3.10. \square

Remark 3.12. By virtue of the holomorphy of $M_1(\cdot)$ in \mathcal{D}_1 and $M_2(\cdot)$ in \mathcal{D}_2 , the identity proved in Theorem 3.11 implies that if λ_0 is a pole of $M_1(\cdot)$, then $1/\bar{\lambda}_0$ is a pole of $M_2(\cdot)$. The same is true for the poles of $M_2(\cdot)$. Thus, the poles of $M_1(\cdot)$ and $M_2(\cdot)$ are symmetric with respect to the unit disc. Hence, $\Lambda_1 = \Lambda_2^\#$.

Definition 3.13. The operator-function defined by the equality

$$(3.26) \quad M(\lambda) = \begin{cases} M_1(\lambda), & \lambda \in \mathcal{D}_1 \\ M_2(\lambda), & \lambda \in \mathcal{D}_2 \end{cases},$$

is called the Weyl function of the operator V , which corresponds to the boundary triplet $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$.

Lemma 3.14. *Let $V : \mathcal{H} \rightarrow \mathcal{H}$ be an isometric operator, and let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet of the isometric operator V . Then:*

(1) *For $\lambda \in \mathcal{D}_1$, the following equality holds*

$$(3.27) \quad \Gamma_2 \left[\begin{array}{c} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{array} \right] = -\frac{1}{\lambda} \gamma_2^\#(\lambda);$$

(2) *for $\lambda \in \mathcal{D}_2$, the following equality holds*

$$(3.28) \quad \Gamma_1 \left[\begin{array}{c} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{array} \right] = \frac{1}{\lambda} \gamma_1^\#(\lambda).$$

Proof. 1) Take $\lambda \in \mathcal{D}_1$, $\mu \in \mathcal{D}_2$ and $h_1 \in \mathfrak{N}_1$. Formula (3.13) yields

$$\widehat{\gamma}_1(\lambda) h_1 - \widehat{\gamma}_1(\mu) h_1 = (\lambda - \mu) \left[\begin{array}{c} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{array} \right] \gamma_1(\mu) h_1.$$

Applying the operator Γ_2 to both sides of the equality, we obtain

$$M_1(\lambda) h_1 - M_1(\mu) h_1 = (\lambda - \mu) \Gamma_2 \left[\begin{array}{c} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{array} \right] \gamma_1(\mu) h_1.$$

In this formula, we replace $M_1(\lambda)$ by $M_2\left(\frac{1}{\lambda}\right)^{[*]}$. In view of formula (3.23), the left side can be written as follows:

$$M_2\left(\frac{1}{\lambda}\right)^{[*]} h_1 - M_1(\mu)^{[*]} h_1 = -\left(1 - \frac{\mu}{\lambda}\right) \gamma_2\left(\frac{1}{\lambda}\right)^{[*]} \gamma_1(\mu) h_1.$$

Equating the right-hand sides of two last formulas, we obtain

$$\Gamma_2 \left[\begin{array}{c} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{array} \right] = -\frac{1}{\lambda} \gamma_2\left(\frac{1}{\lambda}\right)^{[*]}.$$

2) Take $\lambda \in \mathcal{D}_2$, $\mu \in \mathcal{D}_1$, and $h_2 \in \mathfrak{N}_2$. Substituting λ and μ in formula (3.13), we write it in the form

$$\widehat{\gamma}_2(\lambda) h_2 - \widehat{\gamma}_2(\mu) h_2 = (\lambda - \mu) \left[\begin{array}{c} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{array} \right] \gamma_2(\mu) h_2.$$

Applying the operator Γ_1 to both sides of the equality, we obtain

$$M_2(\lambda) h_2 - M_2(\mu) h_2 = (\lambda - \mu) \Gamma_1 \left[\begin{array}{c} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{array} \right] \gamma_2(\mu) h_2.$$

Replacing $M_2(\lambda)$ in this formula by $M_1(1/\overline{\lambda})^{[*]}$, we have

$$M_1(1/\overline{\lambda})^{[*]} - M_2(\mu) h_2 = (\lambda - \mu) \Gamma_1 \left[\begin{array}{c} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{array} \right] \gamma_2(\mu) h_2.$$

In view of formula (3.22), we write the left-hand side as

$$M_1(1/\overline{\lambda})^{[*]} - M_2(\mu) h_2 = \left(1 - \frac{\mu}{\lambda}\right) \gamma_1\left(\frac{1}{\overline{\lambda}}\right)^{[*]} \gamma_2(\mu) h_2.$$

Comparing the right-hand sides of two last formulas, we obtain formula (3.28). \square

3.3. Description of resolvents of extensions of V . Below, we present two theorems, describing the spectrum and the resolvents of extensions V_θ of the operator V . The first theorem gives such a description for the points λ lying outside the unit disc \mathbb{D} , i.e., $\lambda \in \mathcal{D}_1 \subset \mathbb{D}_e$. Recall, that $R_\lambda(T)$ means the resolvent of T at λ (see formula (2.3)).

Theorem 3.15. *Let $V : \mathcal{H} \rightarrow \mathcal{H}$ be an isometric operator, let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet of the isometric operator V , and let θ be a l.r. from \mathfrak{N}_2 to \mathfrak{N}_1 . Then, for $\lambda \in \mathcal{D}_1$ the following assertions are valid:*

(1) $\lambda \in \sigma_p(V_\theta)$ iff $0 \in \sigma_p(\theta^{-1} - M_1(\lambda))$, in this case,

$$\ker(\theta^{-1} - M_1(\lambda)) = \left\{ \Gamma_1 \begin{bmatrix} f \\ \lambda f \end{bmatrix}, f \in \ker(V_\theta - \lambda) \right\}.$$

(2) $\lambda \in \rho(V_\theta) \cap \mathcal{D}_1$ iff $0 \in \rho(\theta^{-1} - M_1(\lambda))$, for $\lambda \in \rho(V_\theta) \cap \mathcal{D}_1$ the resolvent of the extension V_θ can be determined from the formula

$$(3.29) \quad R_\lambda(V_\theta) = R_\lambda(V_1) - \lambda^{-1} \gamma_1(\lambda) (\theta^{-1} - M_1(\lambda))^{-1} \gamma_2^\#(\lambda).$$

Proof. 1) Let $\lambda \in \sigma_p(V_\theta)$ and f_λ be an eigenvector of V_θ corresponding to the eigenvalue λ . Hence,

$$\begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} \in V_\theta, \quad f_\lambda \in \mathfrak{N}_\lambda(V), \quad \text{and} \quad M_1(\lambda) \Gamma_1 \widehat{f}_\lambda = \Gamma_2 \widehat{f}_\lambda.$$

Since $f_\lambda \in \text{dom } V_\theta$, we have $\begin{bmatrix} \Gamma_1 \widehat{f}_\lambda \\ \Gamma_2 \widehat{f}_\lambda \end{bmatrix} \in \theta^{-1}$. Hence, $(\theta^{-1} - M_1(\lambda)) \Gamma_1 \widehat{f}_\lambda = 0$.

Conversely, if $(\theta^{-1} - M_1(\lambda)) h_1 = 0$ for some $h_1 \in \mathfrak{N}_1$, then the vector $f_\lambda := \gamma_1(\lambda) h_1 \in \mathfrak{N}_\lambda(V)$, and, hence, $f_\lambda \in \sigma_p(V_\theta)$.

2) Assume that $0 \in \rho(\theta^{-1} - M_1(\lambda))$, $\begin{bmatrix} f \\ f' \end{bmatrix} \in V_\theta$ and $g \in \mathcal{H}$. Lemma 3.2 implies that the solution of the equation

$$(3.30) \quad f' - \lambda f = g$$

can be presented in the form

$$(3.31) \quad \begin{bmatrix} f \\ f' \end{bmatrix} = \begin{bmatrix} f_1 \\ f'_1 \end{bmatrix} + \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix}, \quad \text{where } \widehat{f}_1 \in V_1, \quad \widehat{f}_\lambda \in \mathfrak{N}_\lambda(V).$$

Then formula (3.30) yields

$$(3.32) \quad f_1 = (V_1 - \lambda)^{-1} g.$$

Applying the operators Γ_1 and Γ_2 to the equality (3.31), we obtain

$$\begin{aligned} \Gamma_1 \widehat{f} &= \Gamma_1 \widehat{f}_\lambda \\ \Gamma_2 \widehat{f} &= \Gamma_2 \begin{bmatrix} (V_1 - \lambda)^{-1} g \\ g + \lambda (V_1 - \lambda)^{-1} g \end{bmatrix} + \Gamma_2 \widehat{f}_\lambda = -\frac{1}{\lambda} \gamma_2^\#(\lambda) g + M_1(\lambda) \Gamma_1 \widehat{f}. \end{aligned}$$

Since $0 \in \rho(\theta^{-1} - M_1(\lambda))$, the previous equality yields

$$(3.33) \quad \begin{aligned} \Gamma_1 \widehat{f}_\lambda &= -\frac{1}{\lambda} (\theta^{-1} - M_1(\lambda))^{-1} \gamma_2^\#(\lambda) g, \\ f_\lambda &= -\frac{1}{\lambda} \gamma_1(\lambda) (\theta^{-1} - M_1(\lambda))^{-1} \gamma_2^\#(\lambda) g. \end{aligned}$$

Equalities (3.31), (3.32), and (3.33) yield equality (3.29).

Conversely, let $\lambda \in \rho(V_\theta)$. By virtue of item 1), to prove the membership $0 \in \rho(\theta^{-1} - M_1(\lambda))$, it is sufficient to show that $\text{ran}(\theta^{-1} - M_1(\lambda)) = \mathfrak{N}_2$. Indeed, by virtue of the surjectivity of the mapping Γ , there exists the vector $\widehat{f}_1 \in V^{-[*]}$ for any $h_2 \in \mathfrak{N}_2$ such that $\Gamma \widehat{f}_1 = \begin{bmatrix} 0 \\ h_2 \end{bmatrix}$. Since $\Gamma_1 \widehat{f}_1 = 0$, we have $\widehat{f}_1 \in V_1$. We set $f = (V_\theta - \lambda)^{-1}(f'_1 - \lambda f_1)$. Then $f_\lambda := f - f_1 \in \mathfrak{N}_\lambda(V)$ and $f = f_1 + f_\lambda$. Since $\begin{bmatrix} \Gamma_1 \widehat{f} \\ \Gamma_2 \widehat{f} \end{bmatrix} = \begin{bmatrix} \Gamma_1 \widehat{f}_\lambda \\ \Gamma_2 \widehat{f} \end{bmatrix} \in \theta^{-1}$, we obtain

$$\Gamma_2 \widehat{f} - M_1(\lambda) \Gamma_1 \widehat{f}_\lambda = \Gamma_2(\widehat{f} - \widehat{f}_\lambda) = \Gamma_2 \widehat{f}_1 = h_2.$$

This proves the equality $\text{ran}(\theta^{-1} - M_1(\lambda)) = \mathfrak{N}_2$ and also the inclusion $0 \in \rho(\theta^{-1} - M_1(\lambda))$. \square

Let θ be a closed l.r. from \mathfrak{N}_2 to \mathfrak{N}_1 . Then there exists a Hilbert space H and bounded operators $K_i : H \rightarrow \mathfrak{N}_i$ $i = 1, 2$, such that

$$(3.34) \quad \theta = \left\{ \begin{bmatrix} K_2 h \\ K_1 h \end{bmatrix} : h \in H \right\}.$$

Corollary 3.16. *If we write the l.r. θ in terms of the operators K_1 and K_2 (see 3.34), then $\lambda \in \rho(V_\theta)$ iff $0 \in \rho(K_2 - M_1(\lambda)K_1)$. Formula (3.29) takes the form*

$$(3.35) \quad R_\lambda(V_\theta) = R_\lambda(V_1) - \lambda^{-1} \gamma_1(\lambda) K_1 (K_2 - M_1(\lambda)K_1)^{-1} \gamma_2^\#(\lambda).$$

Corollary 3.17. *Let θ be the graph of a unitary operator U from \mathfrak{N}_2 to \mathfrak{N}_1 . Then, for $\lambda \in \mathcal{D}_1$ such that $0 \in \rho(I - M_1(\lambda)U)$, we obtain $\lambda \in \rho(V_\theta)$, and the resolvent of an extension V_θ can be found by the formula*

$$(3.36) \quad R_\lambda(V_\theta) = R_\lambda(V_1) - \lambda^{-1} \gamma_1(\lambda) U (I - M_1(\lambda)U)^{-1} \gamma_2^\#(\lambda).$$

The following result for the point λ inside the unit disc \mathbb{D} can be proved analogously.

Theorem 3.18. *Let $V : \mathcal{H} \rightarrow \mathcal{H}$ be an isometric operator, let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet of an isometric operator V , and θ be a l.r. from \mathfrak{N}_2 to \mathfrak{N}_1 . Then, for $\lambda \in \mathcal{D}_2$, the following assertions are true:*

(1) $\lambda \in \sigma_p(V_\theta)$ iff $0 \in \sigma_p(\theta - M_2(\lambda))$, in this case,

$$\ker(\theta - M_2(\lambda)) = \left\{ \Gamma_2 \begin{bmatrix} f \\ \lambda f \end{bmatrix}, f \in \ker(V_\theta - \lambda) \right\}.$$

(2) $\lambda \in \rho(V_\theta)$ iff $0 \in \rho(\theta - M_2(\lambda))$; for $\lambda \in \rho(V_\theta) \cap \mathcal{D}_2$ the resolvent of an extension V_θ can be found by the formula

$$(3.37) \quad R_\lambda(V_\theta) = R_\lambda(V_2) + \lambda^{-1} \gamma_2(\lambda) (\theta - M_2(\lambda))^{-1} \gamma_1^\#(\lambda).$$

Remark 3.19. Note that formula (3.37) can be obtained from formula (3.29) by taking "sharp" of both sides and using the formulas $V_1^{-[*]} = V_2$ and $M_1(\lambda)^\# = M_2(\lambda)$. Indeed,

$$\begin{aligned} R_{1/\bar{\lambda}}(V_\theta)^{[*]} &= R_{1/\bar{\lambda}}(V_1)^{[*]} - \lambda \gamma_2(\lambda) (\theta^{-1} - M_1(1/\bar{\lambda}))^{-[*]} \gamma_1^\#(\lambda), \\ -\lambda I - \lambda^2 R_\lambda(V_{\theta^{-[*]}}) &= -\lambda I - \lambda^2 R_\lambda(V_2) - \lambda \gamma_2(\lambda) (\theta^{-1} - M_1^\#(\lambda))^{-1} \gamma_1^\#(\lambda). \end{aligned}$$

After simplifications one gets

$$R_\lambda(V_{\theta-[*]}) = R_\lambda(V_2) + \lambda^{-1}\gamma_2(\lambda) \left(\theta^{[*]} - M_2(\lambda) \right)^{-1} \gamma_1^\#(\lambda).$$

Corollary 3.20. *If the l.r. θ is written in terms of the operators K_1 and K_2 (see (3.34)), then formula (3.37) takes the form*

$$(3.38) \quad R_\lambda(V_\theta) = R_\lambda(V_2) + \lambda^{-1}\gamma_2(\lambda)K_2(K_1 - M_2(\lambda)K_2)^{-1}\gamma_1^\#(\lambda).$$

4. LINEAR FRACTIONAL TRANSFORMATIONS OF AN ISOMETRIC OPERATOR.

Let $z_0(\in \mathbb{D}_e)$ be a regular type point for an isometric operator V and let \tilde{V} be an extension of V such that $z_0 \in \rho(\tilde{V})$. Then the operators

$$(4.1) \quad V_0 := (I - |z_0|^2)(V - z_0I)^{-1} - \bar{z}_0I$$

$$(4.2) \quad \tilde{V}_0 := (I - |z_0|^2)(\tilde{V} - z_0I)^{-1} - \bar{z}_0I$$

are well defined.

Lemma 4.1. *The resolvent set of \tilde{V}_0 , i.e. the set of $\zeta \in \mathbb{C}$ such that $(\tilde{V}_0 - \zeta I)$ is boundary invertible, is connected to the resolvent set of the l.r. \tilde{V} by the formula*

$$(4.3) \quad \zeta = \frac{1 - \bar{z}_0\lambda}{\lambda - z_0}, \text{ where } \lambda \in \rho(\tilde{V}).$$

Proof. Indeed, the operator

$$(4.4) \quad \begin{aligned} \tilde{V}_0 - \zeta I &= (I - |z_0|^2)(\tilde{V} - z_0I)^{-1} - (\bar{z}_0 + \zeta)I \\ &= \frac{|z_0|^2 - 1}{\lambda - z_0} \left(I - (\lambda - z_0)(\tilde{V} - z_0I)^{-1} \right) \end{aligned}$$

is invertible if and only if $\lambda = \frac{1 + \zeta z_0}{\bar{z}_0 + \zeta} \in \rho(\tilde{V})$. Then $\zeta = \frac{1 - \bar{z}_0\lambda}{\lambda - z_0}$. \square

Lemma 4.2. *The connection between the resolvents of \tilde{V}_0 and \tilde{V} are the following*

$$(4.5) \quad (\tilde{V}_0 - \zeta I)^{-1} = \frac{\lambda - z_0}{|z_0|^2 - 1} \left(I + (\lambda - z_0)(\tilde{V} - \lambda I)^{-1} \right).$$

Proof. Namely it follows from (4.4) that

$$\begin{aligned} (\tilde{V}_0 - \zeta I)^{-1} &= \frac{|z_0|^2 - 1}{\lambda - z_0} \left(I - (\lambda - z_0)(\tilde{V} - z_0I)^{-1} \right)^{-1} \\ &= \frac{\lambda - z_0}{|z_0|^2 - 1} \left(I + (\lambda - z_0)(\tilde{V} - \lambda I)^{-1} \right). \end{aligned} \quad \square$$

In the next Lemma connections between boundary operators, γ -fields and Weyl-functions of V_0 and V will be established.

Lemma 4.3. *Let V be an isometric operator, let $z_0 \in \hat{\rho}(V) \cap \mathbb{D}_e$ and let V_0 and ζ be given by (4.1) and (4.3), and let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet for V . Then:*

(1) *The linear relation $(V_0^{[*]})^{-1}$ takes the form*

$$(4.6) \quad (V_0^{[*]})^{-1} = \left\{ \hat{f} = \begin{bmatrix} h' - z_0 h \\ h - \bar{z}_0 h' \end{bmatrix} : \hat{h} \in V^{-[*]} \right\};$$

(2) A boundary triplet $\Pi^0 = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1^0, \Gamma_2^0\}$ for $V_0^{-[*]}$ can be given by the formulas

$$(4.7) \quad \Gamma_j^0 \widehat{f} = \sqrt{|z_0|^2 - 1} \Gamma_j \widehat{h} \quad (j = 1, 2)$$

where \widehat{f} and \widehat{h} are connected as in (4.6).

(3) The Weyl functions $M_j(\zeta)$ and $M_j^0(\lambda)$ and the γ -fields $\gamma_j(\zeta)$ and $\gamma_j^0(\lambda)$ corresponding to the boundary triplets Π and Π^0 , respectively, are connected by the formulas

$$(4.8) \quad M_j^0(\zeta) = M_j(\lambda), \quad \gamma_j^0(\zeta) = \frac{\lambda - z_0}{\sqrt{|z_0|^2 - 1}} \gamma_j(\lambda) \quad (j = 1, 2).$$

Proof. (1) By using the formula (4.1) one obtains

$$(4.9) \quad V_0^{[*]} = (1 - |z_0|^2) \left(V^{[*]} - \bar{z}_0 \right)^{-1} - z_0 I.$$

The formula (4.6) follows from (4.9).

(2) Next it follows from (4.7), (3.1), and (4.6) that

$$\begin{aligned} & \left[\Gamma_1^0 \widehat{f}, \Gamma_1^0 \widehat{f} \right]_{\mathfrak{N}_1} - \left[\Gamma_2^0 \widehat{f}, \Gamma_2^0 \widehat{f} \right]_{\mathfrak{N}_2} = (|z_0|^2 - 1) \left(\left[\Gamma_1 \widehat{h}, \Gamma_1 \widehat{h} \right]_{\mathfrak{N}_1} - \left[\Gamma_2 \widehat{h}, \Gamma_2 \widehat{h} \right]_{\mathfrak{N}_2} \right) \\ & = (|z_0|^2 - 1) ([h', h']_{\mathcal{H}} - [h, h]_{\mathcal{H}}) = [h - \bar{z}_0 h', h - \bar{z}_0 h']_{\mathcal{H}} - [h' - z_0 h, h' - z_0 h]_{\mathcal{H}} \\ & = [f', f']_{\mathcal{H}} - [f, f]_{\mathcal{H}}. \end{aligned}$$

(3) Let $\widehat{h} = \begin{bmatrix} h \\ h' \end{bmatrix} \in \mathfrak{N}_\lambda(V)$, i.e., $\widehat{h} \in V^{-[*]}$ and $h' = \lambda h$. Then

$$(4.10) \quad \widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix} := \begin{bmatrix} h' - z_0 h \\ h - \bar{z}_0 h' \end{bmatrix} \in \mathfrak{N}_\zeta(V_0).$$

Let us set $u_j := \Gamma_j \widehat{h}$ ($j = 1, 2$). Then by (3.12) $\widehat{\gamma}_j(\lambda) u_j = \widehat{h}$. It follows from (4.7) that

$$\Gamma_j^0 \widehat{f} = \sqrt{|z_0|^2 - 1} \Gamma_j \widehat{h} = \sqrt{|z_0|^2 - 1} u_j$$

and $\widehat{\gamma}_j^0(\zeta) u_j = \frac{1}{\sqrt{|z_0|^2 - 1}} \widehat{h}$. Hence by (3.12) and (4.10)

$$\gamma_j^0(\zeta) u_j = \frac{1}{\sqrt{|z_0|^2 - 1}} \pi_1 \widehat{f} = \frac{\lambda - z_0}{\sqrt{|z_0|^2 - 1}} h = \frac{\lambda - z_0}{\sqrt{|z_0|^2 - 1}} \gamma_j(\lambda) u_j.$$

Since (3.18) and (3.19) one gets

$$M_j^0(\zeta) u_j = \Gamma_i^0 \widehat{\gamma}_j^0(\zeta) u_j = \frac{1}{\sqrt{|z_0|^2 - 1}} \Gamma_i^0 \widehat{f} = \Gamma_i \widehat{h} = \Gamma_i \widehat{\gamma}_j(\lambda) u_j = M_j(\lambda) u_j,$$

where $i, j = 1, 2$ and $i \neq j$. \square

5. DESCRIPTION OF GENERALIZED RESOLVENTS.

Definition 5.1. (see [16]) The operator-function \mathbb{R}_λ holomorphic in neighborhood \mathcal{O} of the point $\zeta \in \mathcal{D}_1$ is called the generalized resolvent of an isometric operator $V : \mathcal{H} \rightarrow \mathcal{H}$, if there exist a Pontryagin space $\widetilde{\mathcal{H}} \supset \mathcal{H}$ and a unitary extension $\widetilde{V} : \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$ of the operator V such that $\lambda \in \rho(\widetilde{V})$, and if the equality

$$(5.1) \quad \mathbf{R}_\lambda = P_{\mathcal{H}} \left(\widetilde{V} - \lambda \right)^{-1} \upharpoonright \mathcal{H}, \quad \lambda \in \rho(\widetilde{V}) \cap \mathcal{O}$$

in which $P_{\mathcal{H}}$ is the orthoprojector from $\tilde{\mathcal{H}}$ onto \mathcal{H} holds.

Definition 5.2. A unitary extension \tilde{V} of an operator V is called minimal, if $\mathcal{H}_{\tilde{V}} = \tilde{\mathcal{H}}$, where

$$(5.2) \quad \mathcal{H}_{\tilde{V}} := \overline{\text{span}} \left\{ \mathcal{H} + (\tilde{V} - \lambda)^{-1} \mathcal{H} : \lambda \in \rho(\tilde{V}) \right\}.$$

Definition 5.3. A unitary extension \tilde{V} of an operator V is called regular, if $\tilde{\mathcal{H}}_{\tilde{V}}^{[\perp]} := \tilde{\mathcal{H}}[-]\mathcal{H}_{\tilde{V}}$ is a Hilbert space.

Proposition 5.4. Let a regular extension \tilde{V} of an operator V be not minimal. Then the following decompositions are valid:

$$(5.3) \quad \tilde{\mathcal{H}} = \mathcal{H}_{\tilde{V}}^{[\perp]}[+]\mathcal{H}_{\tilde{V}} \text{ and } \tilde{V} = \tilde{V}_u[+]\tilde{V}_m.$$

Here \tilde{V}_m is the minimal extension of the operator V and \tilde{V}_u is a unitary operator in a Hilbert space $\mathcal{H}_{\tilde{V}}^{[\perp]}$. In this case,

$$(5.4) \quad P_{\mathcal{H}}(\tilde{V} - \lambda)^{-1} \upharpoonright \mathcal{H} = P_{\mathcal{H}}(\tilde{V}_m - \lambda)^{-1} \upharpoonright \mathcal{H}.$$

Proof. Since \tilde{V} is the regular extension, we have $\text{ind } _ \mathcal{H}_{\tilde{V}} = \text{ind } _ \tilde{\mathcal{H}} = \tilde{\kappa}$. Hence, $\mathcal{H}_{\tilde{V}}$ is not degenerate.

We now show that $\mathcal{H}_{\tilde{V}}$ and $\mathcal{H}_{\tilde{V}}^{[\perp]}$ are invariant for \tilde{V} . Let us take different λ_1 and λ_2 from $\rho(\tilde{V})$. Let $h \in \mathcal{H}$. Then $u := (\tilde{V} - \lambda_2)^{-1}h \in \mathcal{H}_{\tilde{V}}$. Let the operator $(\tilde{V} - \lambda_1)^{-1}$ act on this vector:

$$(\tilde{V} - \lambda_1)^{-1}(\tilde{V} - \lambda_2)^{-1}h = \frac{1}{\lambda_1 - \lambda_2} \left((\tilde{V} - \lambda_1)^{-1} - (\tilde{V} - \lambda_2)^{-1} \right) h \in \mathcal{H}_{\tilde{V}}.$$

The case where λ_1 and λ_2 coincide with each other follows from the previous one, if λ_1 tends to λ_2 .

Consider now the vectors $v \in \mathcal{H}_{\tilde{V}}^{[\perp]}$ and $u \in \mathcal{H}_{\tilde{V}}$. Then

$$\left[(\tilde{V} - \lambda)^{-1}v, u \right]_{\tilde{\mathcal{H}}} = \left[v, (\tilde{V}^* - \bar{\lambda})^{-1}u \right]_{\tilde{\mathcal{H}}} = \left[v, \frac{1}{\bar{\lambda}} \left(-I + (I - \bar{\lambda}\tilde{V})^{-1} \right) u \right]_{\tilde{\mathcal{H}}} = 0.$$

Here, we use the fact that, for the unitary operator \tilde{V} , the inclusion $\lambda \in \rho(\tilde{V})$ yields the inclusion $\frac{1}{\bar{\lambda}} \in \rho(\tilde{V})$.

Thus, $\tilde{\mathcal{H}} = \mathcal{H}_{\tilde{V}}^{[\perp]}[+]\mathcal{H}_{\tilde{V}}$ and $\tilde{V} = \begin{bmatrix} \tilde{V}_u & 0 \\ 0 & \tilde{V}_m \end{bmatrix}$, where \tilde{V}_m is the minimal extension of the operator V in $\mathcal{H}_{\tilde{V}}$.

The equality

$$P_{\mathcal{H}}(\tilde{V} - \lambda)^{-1} \upharpoonright \mathcal{H} = P_{\mathcal{H}}(\tilde{V}_m - \lambda)^{-1} \upharpoonright \mathcal{H},$$

follows from representation (5.3). \square

Theorem 5.5. Let V be an isometry in a Pontryagin space \mathcal{H} with negative index κ , let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet for V , $V_i = \ker \Gamma_i$, and $\gamma_i(\cdot)$, $M_i(\cdot)$, $i = 1, 2$ be the corresponding γ -fields and the Weyl functions and the condition (A) holds.

Let $\tilde{\mathcal{H}} = \mathcal{H}^{[\perp]}[+]\mathcal{H}$ be a Pontryagin space $\text{ind } _ \tilde{\mathcal{H}} = \tilde{\kappa} \geq \kappa$, $\text{ind } _ \mathcal{H}^{[\perp]} = \tilde{\kappa} - \kappa$. We define the projectors π_1 and π_2 from $\mathcal{H}^{[\perp]} \times \mathcal{H}^{[\perp]}$ onto the first and second components $\mathcal{H}^{[\perp]} \times \mathcal{H}^{[\perp]}$,

$$\pi_1 \widehat{h} = h, \quad \pi_2 \widehat{h} = h', \quad \text{where } \widehat{h} = \begin{bmatrix} h \\ h' \end{bmatrix} \in (\mathcal{H}^{[\perp]})^2.$$

Then

(1) the adjoint l.r. for V^{-1} in the space $\widetilde{\mathcal{H}}$ takes the form

$$(5.5) \quad V_{\widetilde{\mathcal{H}}}^{-[*]} = (\mathcal{H}^{[\perp]})^2[+]V^{-[*]};$$

(2) the operators

$$(5.6) \quad \widetilde{\Gamma}_1 = \begin{bmatrix} \pi_2 & 0 \\ 0 & \Gamma_1 \end{bmatrix} \in [(\mathcal{H}^{[\perp]})^2[+]V^{-[*]}, \mathcal{H}^{[\perp]}[+]\mathfrak{N}_1].$$

$$(5.7) \quad \widetilde{\Gamma}_2 = \begin{bmatrix} \pi_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \in [(\mathcal{H}^{[\perp]})^2[+]V^{-[*]}, \mathcal{H}^{[\perp]}[+]\mathfrak{N}_2].$$

are the boundary operators in the boundary triplet

$$\widetilde{\Pi} = \{(\mathcal{H}^{[\perp]}[+]\mathfrak{N}_1) \oplus (\mathcal{H}^{[\perp]}[+]\mathfrak{N}_2), \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$$

for the isometry V in $\widetilde{\mathcal{H}}$.

Moreover,

$$(5.8) \quad \begin{aligned} \widetilde{V}_1 (= \ker \widetilde{\Gamma}_1) &= (\mathcal{H}^{[\perp]} \oplus \{0\})[+]V_1, \\ \widetilde{V}_2 (= \ker \widetilde{\Gamma}_2) &= (\{0\} \oplus \mathcal{H}^{[\perp]})[+]V_2 \end{aligned}$$

and the corresponding γ -fields and the Weyl functions for the boundary triplet $\widetilde{\Pi}$ take the form

$$(5.9) \quad \begin{aligned} \widetilde{\gamma}_1(\lambda) &= \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^{[\perp]}} & 0 \\ 0 & \gamma_1(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_1 \\ \widetilde{\gamma}_2(\lambda) &= \begin{bmatrix} I_{\mathcal{H}^{[\perp]}} & 0 \\ 0 & \gamma_2(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_2 \end{aligned}$$

$$(5.10) \quad \begin{aligned} \widetilde{M}_1(\lambda) &= \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^{[\perp]}} & 0 \\ 0 & M_1(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_1 \\ \widetilde{M}_2(\lambda) &= \begin{bmatrix} \lambda I_{\mathcal{H}^{[\perp]}} & 0 \\ 0 & M_2(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_2 \end{aligned}$$

Proof. Equality (5.5) is obvious. Let us prove equalities (5.6) and (5.7). Suppose $\widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix}$, $\widehat{g} = \begin{bmatrix} g \\ g' \end{bmatrix} \in V^{-[*]}$ and $\widehat{m} = \begin{bmatrix} m \\ m' \end{bmatrix}$, $\widehat{n} = \begin{bmatrix} n \\ n' \end{bmatrix} \in (\mathcal{H}^{[\perp]})^2$. Then $\widehat{f} + \widehat{m}$ and $\widehat{g} + \widehat{n} \in V^{-[*]}[+](\mathcal{H}^{[\perp]})^2$. Let us check the general Green equality

$$\begin{aligned} &[f', g']_{\mathcal{H}} - [f, g]_{\mathcal{H}} + [m', n']_{\mathcal{H}^{[\perp]}} - [m, n]_{\mathcal{H}^{[\perp]}} \\ &= [\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g}]_{\mathfrak{N}_1} - [\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g}]_{\mathfrak{N}_2} + [\pi_2 \widehat{m}, \pi_2 \widehat{n}]_{\mathcal{H}^{[\perp]}} - [\pi_1 \widehat{m}, \pi_1 \widehat{n}]_{\mathcal{H}^{[\perp]}} \\ &= [\widetilde{\Gamma}_1(\widehat{f} + \widehat{m}), \widetilde{\Gamma}_1(\widehat{g} + \widehat{n})]_{\mathcal{H}^{[\perp]}[+]\mathfrak{N}_1} - [\widetilde{\Gamma}_2(\widehat{f} + \widehat{m}), \widetilde{\Gamma}_2(\widehat{g} + \widehat{n})]_{\mathcal{H}^{[\perp]}[+]\mathfrak{N}_2}. \end{aligned}$$

The defect space takes the form

$$\widehat{\mathfrak{N}}_{\lambda}(V)_{\widetilde{\mathcal{H}}} = \left\{ \widehat{f}_{\lambda} + \widehat{g} : \widehat{f}_{\lambda} \in \widehat{\mathfrak{N}}_{\lambda}(V), \widehat{g} = \begin{bmatrix} g \\ \lambda g \end{bmatrix} \in \begin{bmatrix} \mathcal{H}^{[\perp]} \\ \mathcal{H}^{[\perp]} \end{bmatrix} \right\}.$$

Formulas (5.8) have become obvious now. Prove the formulas of γ -fields. Taking into account

$$\tilde{\Gamma}_1(\hat{f}_\lambda + \hat{g}) = \lambda g + \Gamma_1 \hat{f}_\lambda,$$

we obtain

$$\tilde{\gamma}_1(\lambda) = \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^{[\perp]}} & 0 \\ 0 & \gamma_1(\lambda) \end{bmatrix}.$$

Similarly, we get the formula of $\tilde{\gamma}_2(\cdot)$ from

$$\tilde{\Gamma}_2(\hat{f}_\lambda + \hat{g}) = g + \Gamma_2 \hat{f}_\lambda.$$

Finally let us prove the formulas for Weyl functions. Take $h_1 \in \mathfrak{N}_1$ and $g \in \mathcal{H}^{[\perp]}$. Then by definition of Weyl function one gets

$$\tilde{M}_1(\lambda)(h_1 + g) = \tilde{\Gamma}_2 \tilde{\gamma}_1(\lambda)(h_1 + g),$$

where $\tilde{\gamma}_1(\lambda) = \begin{bmatrix} \tilde{\gamma}_1(\lambda) \\ \lambda \tilde{\gamma}_1(\lambda) \end{bmatrix}$. We obtain

$$\tilde{M}_1(\lambda)(h_1 + g) = \tilde{\Gamma}_2 \left(\begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^{[\perp]}} \\ I_{\mathcal{H}^{[\perp]}} \end{bmatrix} g + \hat{\gamma}_1(\lambda) h_1 \right) = \frac{1}{\lambda} g + M_1(\lambda) h_1.$$

Similarly, we obtain the latter formula. Now suppose $h_2 \in \mathfrak{N}_2$. Then

$$\begin{aligned} \tilde{M}_2(\lambda)(h_2 + g) &= \tilde{\Gamma}_1 \hat{\gamma}_2(\lambda)(h_2 + g) = \tilde{\Gamma}_1 \left(\begin{bmatrix} I_{\mathcal{H}^{[\perp]}} \\ \lambda I_{\mathcal{H}^{[\perp]}} \end{bmatrix} g + \hat{\gamma}_2(\lambda) h_2 \right) \\ &= \lambda g + M_2(\lambda) h_2. \end{aligned} \quad \square$$

We recall the basic notions of the theory of unitary colligations (see [1], [7]). Let $\mathcal{H}, \mathfrak{N}_2$ and \mathfrak{N}_1 be Pontryagin spaces, and let $U = \begin{bmatrix} T & F \\ G & H \end{bmatrix}$ be a unitary operator from $\mathcal{H}[\perp]\mathfrak{N}_2$ to $\mathcal{H}[\perp]\mathfrak{N}_1$. Then the quadruple $\Delta = (\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1; U)$ is called a unitary colligation. The spaces $\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1$ are called, respectively, the space of states, space of inputs, and space of outputs, and the operator U is called the connecting operator of the colligation Δ .

The colligation Δ is called simple, if there exists no subspace in the space \mathcal{H} reducing U . The operator-function

$$(5.11) \quad \Theta(\lambda) = H + \lambda G(I - \lambda T)^{-1} F : \mathfrak{N}_2 \rightarrow \mathfrak{N}_1 \quad (\lambda^{-1} \in \rho(T))$$

is called the characteristic function of a colligation Δ (or the scattering matrix of the unitary operator U relative to the channel spaces \mathfrak{N}_2 and \mathfrak{N}_1 in the case where $\mathfrak{N}_2, \mathfrak{N}_1, \mathcal{H}$ are Hilbert ones [2]). The characteristic function characterizes a simple unitary colligation to within a unitary equivalence.

We recall that for the components of a unitary colligation the following relations

$$(5.12) \quad \begin{aligned} T^{[*]}T + G^{[*]}G &= I_{\mathcal{H}}, \quad F^{[*]}F + H^{[*]}H = I_{\mathfrak{N}_2}, \quad T^{[*]}F + G^{[*]}H = 0, \\ TT^{[*]} + FF^{[*]} &= I_{\mathcal{H}}, \quad GG^{[*]} + HH^{[*]} = I_{\mathfrak{N}_1}, \quad TG^{[*]} + FH^{[*]} = 0 \end{aligned}$$

hold.

Proposition 5.6. [9] *Let $\Delta = (\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$ be a unitary colligation and $\Theta(\cdot)$ be the characteristic function of this colligation. Then*

$$(5.13) \quad \Theta(\lambda) = P_{\mathfrak{N}_1}(I - \lambda U P_{\mathcal{H}})^{-1} U \upharpoonright \mathfrak{N}_2 = P_{\mathfrak{N}_1} U (I - \lambda P_{\mathcal{H}} U)^{-1} \upharpoonright \mathfrak{N}_2,$$

where $P_{\mathcal{H}}$ and $P_{\mathfrak{N}_i}$ are orthoprojections from $\mathcal{H}[\perp]\mathfrak{N}_i$ onto \mathcal{H} and \mathfrak{N}_i ($i = 1, 2$), respectively.

Proof. Indeed, by the equality

$$(I_{\mathcal{H} \oplus \mathfrak{N}_2} - \lambda P_{\mathcal{H}} U)^{-1} = \begin{bmatrix} (I_{\mathcal{H}} - \lambda T)^{-1} & \lambda(I_{\mathcal{H}} - \lambda T)^{-1} F \\ 0 & I_{\mathfrak{N}_2} \end{bmatrix},$$

we get

$$(5.14) \quad U(I - \lambda P_{\mathcal{H}} U)^{-1} = \begin{bmatrix} T(I - \lambda T)^{-1} & F + \lambda T(I - \lambda T)^{-1} F \\ G(I - \lambda T)^{-1} & H + \lambda G(I - \lambda T)^{-1} F \end{bmatrix}$$

and the bottom right corner coincides with $\Theta(\lambda)$. This proves the first equation in (5.13).

Further, note that $U^{-1} = U^{[*]}$ and

$$U(I - \lambda P_{\mathcal{H}} U)^{-1} = (U^{-1} - \lambda P_{\mathcal{H}})^{-1} = (U^{-1}(I - \lambda U P_{\mathcal{H}}))^{-1} = (1 - \lambda U P_{\mathcal{H}})^{-1} U,$$

we obtain the second and the third equalities for $\Theta(\lambda)$. \square

Theorem 5.7. *Let V be an isometric operator in \mathcal{H} , let $\tilde{\mathcal{H}} = \mathcal{H}[\dot{+}] \mathcal{H}^{[\perp]}$ be a Pontryagin space with negative index $\text{ind}_- \tilde{\mathcal{H}} = \tilde{\kappa}$, $\text{ind}_- \mathcal{H} = \kappa$, and let $\tilde{\Pi}$ be the boundary triplet constructed in Theorem 5.5. Then:*

- (1) *Any unitary extension $\tilde{V} \in \mathcal{C}(\tilde{\mathcal{H}})$ of the operator V can be represented in the form $\tilde{V} = \tilde{V}_\theta := \tilde{\Gamma}^{-1} \theta^{-1}$, where θ is the graph of the unitary operator*

$$(5.15) \quad U = \begin{bmatrix} T & F \\ G & H \end{bmatrix} : \begin{bmatrix} \mathcal{H}^{[\perp]} \\ \mathfrak{N}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}^{[\perp]} \\ \mathfrak{N}_1 \end{bmatrix}.$$

- (2) *A unitary extension $\tilde{V}_\theta \in \mathcal{C}(\tilde{\mathcal{H}})$ of the operator V is minimal iff the unitary colligation $\Delta = (\mathcal{H}^{[\perp]}, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$ is simple.*
(3) *If $\Theta(\lambda)$ is the characteristic function of the unitary colligation $\Delta = (\mathcal{H}^{[\perp]}, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$, then the generalized resolvent of the operator V , which corresponds to the extension \tilde{V}_θ , takes the following form for $\lambda \in \rho(\tilde{V}) \cap \mathcal{D}_1$:*

$$(5.16) \quad \mathbf{R}_\lambda = R_\lambda(V_1) - \lambda^{-1} \gamma_1(\lambda) \Theta(1/\lambda) (I - M_1(\lambda) \Theta(1/\lambda))^{-1} \gamma_2^\#(\lambda);$$

and if $\lambda \in \rho(\tilde{V}) \cap \mathcal{D}_2$, it takes the form

$$(5.17) \quad \mathbf{R}_\lambda = R_\lambda(V_2) + \lambda^{-1} \gamma_2(\lambda) \Theta(\bar{\lambda})^{[*]} \left(I - M_2(\lambda) \Theta(\bar{\lambda})^{[*]} \right)^{-1} \gamma_1^\#(\lambda).$$

Proof. 1) The assertion of this item of the theorem is a consequence of Theorem 3.3.3 and the remark after it.

2) Let the colligation $\Delta = (\mathcal{H}^{[\perp]}, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$ be not simple, i.e., $\mathcal{H}^{[\perp]} = \mathcal{H}_1^{[\perp]}[\dot{+}] \mathcal{H}_2^{[\perp]}$. Then the unitary operator U takes the form $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} :$

$\begin{bmatrix} \mathcal{H}_1^{[\perp]} \\ \mathcal{H}_2^{[\perp]}[\dot{+}] \mathfrak{N}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1^{[\perp]} \\ \mathcal{H}_2^{[\perp]}[\dot{+}] \mathfrak{N}_1 \end{bmatrix}$. In view of operators $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ (see formulas 5.6)

and 5.7), we can conclude that they act from $\begin{bmatrix} \mathcal{H}_1^{[\perp]} \\ \mathcal{H}_2^{[\perp]} \end{bmatrix}$ to $\mathcal{H}_1^{[\perp]}$ as projections. Hence,

$\tilde{V} = V_\theta$ will have a reducing subspace, namely, $\mathcal{H}_1^{[\perp]}$. Thus, \tilde{V} is not the minimal extension of the operator V in $\tilde{\mathcal{H}}$. The proof of this assertion in the reverse direction is analogous. reverse direction is analogous.

3) Using formulas (3.35) and (3.38) for the resolvents of extensions of the operator V , we now find the resolvent of the unitary extension $\tilde{V} = \tilde{V}_\theta : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$, where

$$\theta = \left\{ \begin{bmatrix} h \\ Uh \end{bmatrix} : h \in \begin{bmatrix} \mathcal{H}^{[\perp]} \\ \mathfrak{N}_2 \end{bmatrix} \right\}.$$

Then with regard for (5.1), we obtain

$$\begin{aligned} \mathbf{R}_\lambda g &= P_{\mathcal{H}} R_\lambda(\tilde{V}_1)g - \frac{1}{\lambda} P_{\mathcal{H}} \tilde{\gamma}_1(\lambda) U \left(I - \tilde{M}_1(\lambda) U \right)^{-1} \tilde{\gamma}_2^\#(\lambda) g \\ (5.18) \quad &= R_\lambda(V_1)g - \frac{1}{\lambda} \gamma_1(\lambda) P_{\mathfrak{N}_1} U \left(I - \tilde{M}_1(\lambda) U \right)^{-1} \gamma_2^\#(\lambda) g, \end{aligned}$$

where $\lambda \in \rho(\tilde{V}_\theta) \cap \mathcal{D}_1$ and $g \in \mathcal{H}$;

$$\begin{aligned} \mathbf{R}_\lambda g &= P_{\mathcal{H}} R_\lambda(\tilde{V}_2)g + \frac{1}{\lambda} P_{\mathcal{H}} \tilde{\gamma}_2(\lambda) U^{[*]} \left(I - \tilde{M}_2(\lambda) U^{[*]} \right)^{-1} \tilde{\gamma}_1^\#(\lambda) g \\ &= R_\lambda(V_2)g + \frac{1}{\lambda} \gamma_2(\lambda) P_{\mathfrak{N}_1} U^{[*]} \left(I - \tilde{M}_2(\lambda) U^{[*]} \right)^{-1} \gamma_1^\#(\lambda) g, \end{aligned}$$

where $\lambda \in \rho(\tilde{V}_\theta) \cap \mathcal{D}_2$ and $g \in \mathcal{H}$.

The latter formula includes $U^{[*]}$, since

$$\theta = \left\{ \begin{bmatrix} h \\ Uh \end{bmatrix}, h \in \begin{bmatrix} \mathcal{H}^{[\perp]} \\ \mathfrak{N}_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} U^{[*]}g \\ g \end{bmatrix}, g \in \begin{bmatrix} \mathcal{H}^{[\perp]} \\ \mathfrak{N}_1 \end{bmatrix} \right\}$$

by virtue of the unitarity of the operator U .

Using the Frobenius formula for the inverse block matrix, we transform the former formula as

$$\begin{aligned} \left(I - \tilde{M}_1(\lambda) U \right)^{-1} &= \left(I - \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^\perp} & 0 \\ 0 & M_1(\lambda) \end{bmatrix} \begin{bmatrix} T & F \\ G & H \end{bmatrix} \right)^{-1} \\ (5.19) \quad &= \begin{bmatrix} I - \frac{1}{\lambda} T & -\frac{1}{\lambda} F \\ -M_1(\lambda) G & I - M_1(\lambda) H \end{bmatrix}^{-1} \\ &= \begin{bmatrix} * & \frac{1}{\lambda} (I - \frac{1}{\lambda} T)^{-1} F \Phi(\frac{1}{\lambda}) \\ * & \Phi(\frac{1}{\lambda}) \end{bmatrix}, \end{aligned}$$

where $*$ stands for the blocks, which are insignificant, and

$$\begin{aligned} \Phi\left(\frac{1}{\lambda}\right) &:= \left(I - M_1(\lambda) H - \frac{1}{\lambda} M_1(\lambda) G \left(I - \frac{1}{\lambda} T \right)^{-1} F \right)^{-1} \\ (5.20) \quad &= \left(I - M_1(\lambda) \Theta\left(\frac{1}{\lambda}\right) \right)^{-1}. \end{aligned}$$

Substituting (5.19) and (5.20) in (5.18), we obtain

$$\begin{aligned} P_{\mathfrak{N}_1} U \left(I - \tilde{M}_1(\lambda) U \right)^{-1} \upharpoonright_{\mathfrak{N}_2} &= \frac{1}{\lambda} G \left(I - \frac{1}{\lambda} T \right)^{-1} F \Phi\left(\frac{1}{\lambda}\right) + H \Phi\left(\frac{1}{\lambda}\right) \\ &= \Theta\left(\frac{1}{\lambda}\right) \Phi\left(\frac{1}{\lambda}\right) = \Theta\left(\frac{1}{\lambda}\right) \left(I - M_1(\lambda) \Theta\left(\frac{1}{\lambda}\right) \right)^{-1}. \end{aligned}$$

The latter formula is gotten by using Remark 3.19 and formula $\Theta^\#(\frac{1}{\lambda}) = \Theta(\bar{\lambda})^{[*]}$. \square

Theorem 5.8. *Let V be an isometric operator in \mathcal{H} and $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet for it such that condition (A) holds. Then formulas (5.16) and (5.17) establish the bijective correspondence between the set of generalized resolvents of the operator V that correspond to regular extensions $\tilde{V} : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ and the set of operator-functions $\varepsilon(\cdot) \in S_{\tilde{\kappa}-\kappa}(\mathfrak{N}_2, \mathfrak{N}_1)$.*

Proof. Step 1. By virtue of Proposition 5.4 for any general extension \tilde{V} of V exists a minimal extension \tilde{V}_m , such that the generalized resolvent for \mathbf{R}_λ that corresponds to \tilde{V} admits the following representation $\mathbf{R}_\lambda = P_{\mathcal{H}} \left(\tilde{V}_m - \lambda \right)^{-1} \upharpoonright \mathcal{H}$. Then from Theorem 5.7 the operator \tilde{V}_m has the form $\tilde{V}_m = \tilde{\Gamma}^{-1} \theta^{-1}$, where θ is the graph of a unitary operator U from (5.15) and the unitary colligation $\{U, \mathcal{H}^{[\perp]}[\dot{+}] \mathfrak{N}_2, \mathcal{H}^{[\perp]}[\dot{+}] \mathfrak{N}_1\}$ is simple. Let $\Theta(\cdot)$ be the characteristic function for this colligation. Using Theorem 5.7.3 we get formulas (5.16), (5.17). By Theorem 2.5.10 [1] the operator-function $\Theta(\cdot) \in S_{\tilde{\kappa}-\kappa}(\mathfrak{N}_2, \mathfrak{N}_1)$.

To prove the opposite statement we consider separately two cases: when $\varepsilon(\cdot)$ is holomorphic at zero and when it is not.

Step 2. Assume that $\varepsilon(\cdot) \in S_{\tilde{\kappa}-\kappa}(\mathfrak{N}_2, \mathfrak{N}_1)$ and $\varepsilon(\cdot)$ is holomorphic at zero. Then there exists the unique simple unitary colligation to within a unitary equivalence $\Delta = \{\mathcal{H}^{[\perp]}, \mathfrak{N}_2, \mathfrak{N}_1; U\}$ such that $\mathcal{H}^{[\perp]}$ is a Pontryagin space with the index $\tilde{\kappa} - \kappa$ and the characteristic function of this colligation coincide with $\varepsilon(\cdot)$ (see [1, Theorem 2.5.10]).

Consider the boundary triplet from Theorem 5.5. The extension \tilde{V}_U is minimal. By Theorem 5.7.3, the generalized resolvent that corresponds to \tilde{V}_U can be found by (5.16), (5.17).

Step 3. Assume that $\varepsilon(\cdot) \in S_{\tilde{\kappa}-\kappa}(\mathfrak{N}_2, \mathfrak{N}_1)$ and $\varepsilon(\cdot)$ is not holomorphic at zero. Then there exists a point $z_0 \in \mathbb{D}_e$ such that $\varepsilon(\cdot)$ is holomorphic at $1/z_0$. Consider a new operator function $\varepsilon_0(1/\zeta) := \varepsilon(\frac{\zeta + \bar{z}_0}{1 + z_0 \zeta})$, which is holomorphic at zero. By [1, Theorem 2.5.10] there exists the unique simple unitary colligation $\Delta_0 = \{\mathcal{H}^{[\perp]}, \mathfrak{N}_2, \mathfrak{N}_1; U_0\}$ such that the characteristic function of this colligation coincides with $\varepsilon_0(\zeta)$. Consider the isometric operator V_0 defined by (4.2) and the boundary triplet Π_0 for V_0 constructed in Lemma 4.3. Then by the reasoning of Step 2 there exists a unitary extension V_{U_0} of the operator V_0 corresponding to the unitary colligation $\Delta_0 = \{\mathcal{H}^{[\perp]}, \mathfrak{N}_2, \mathfrak{N}_1; U_0\}$ via Theorem 5.5.

The generalized resolvent of V_0 corresponding to this extension \tilde{V}_{U_0} is given by (5.16) for $\zeta \in \mathbb{D}_e$.

$$(5.21) \quad \mathbf{R}_\zeta^0 = R_\zeta(V_1^0) - \zeta^{-1} \gamma_1^0(\zeta) \varepsilon_0(1/\zeta) (I - M_1^0(\zeta) \varepsilon_0(1/\zeta))^{-1} \gamma_2^{0\#}(\zeta).$$

We define a l.r.

$$\tilde{V}_\varepsilon = (I + z_0 \tilde{V}_{U_0})(\tilde{V}_{U_0} + \bar{z}_0)^{-1}.$$

Using formula (5.21) and connections between resolvents, γ -fields and Weyl-functions of \tilde{V}_ε and \tilde{V}_{U_0} (see formula (4.5) and Lemma (4.3)), one gets

$$\begin{aligned} \mathbf{R}_\zeta^0 - R_\zeta(V_1^0) &= \frac{\lambda - z_0}{|z_0|^2 - 1} [I + (\lambda - z_0) \mathbf{R}_\lambda - (I + (\lambda - z_0) R_\lambda(V_1))] \\ &= -\frac{\lambda - z_0}{1 - \bar{z}_0 \lambda} \frac{\lambda - z_0}{\sqrt{|z_0|^2 - 1}} \gamma_1(\lambda) \varepsilon(1/\lambda) (I - M_1(\lambda) \varepsilon(1/\lambda))^{-1} \frac{1/\lambda - \bar{z}_0}{\sqrt{|z_0|^2 - 1}} \gamma_2^\#(\lambda). \end{aligned}$$

After simplifications we obtain the formula

$$\mathbf{R}_\lambda = R_\lambda(V_1) - \lambda^{-1}\gamma_1(\lambda)\varepsilon(1/\lambda)(I - M_1(\lambda)\varepsilon(1/\lambda))^{-1}\gamma_2^\#(\lambda),$$

which proves (5.16). Now by taking "sharp" of both sides we can get formula (5.17) (see Remark 3.19 and the last part of Theorem 5.7 proof). \square

Corollary 5.9. *Let the assumptions of Theorem 5.7 be satisfied. Then formulas for generalized coresolvents have the following forms:*

$$P_{\mathcal{H}}(I - \lambda\tilde{V}) \upharpoonright \mathcal{H} = (I - \lambda V_1)^{-1} + \gamma_1(1/\lambda)\Theta(\lambda)(I - M_1(1/\lambda)\Theta(\lambda))^{-1}\gamma_2(\lambda)^{[*]}$$

for $\lambda \in \rho(\tilde{V})^{-1} \cap \mathcal{D}_2$;

$$P_{\mathcal{H}}(I - \lambda\tilde{V}) \upharpoonright \mathcal{H} = (I - \lambda V_2)^{-1} - \gamma_2(1/\lambda)\Theta^\#(\lambda)(I - M_2(1/\lambda)\Theta^\#(\lambda))^{-1}\gamma_1(\lambda)^{[*]}$$

for $\lambda \in \rho(\tilde{V})^{-1} \cap \mathcal{D}_1$.

REFERENCES

- [1] Alpay D., Dijksma A., Rovnyak J., and de Snoo H.S.V. *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory: Adv. Appl., **96**, Birkhäuser Verlag, Basel, 1997. – 229 p.
- [2] Arov, D. Z. and Grossman, L. Z., The scattering matrices in the theory of the extensions of isometric operators, DAN USSR, **33**, No. 4 (1983), 17–20.
- [3] Azizov, T. Ya. and Iokhvidov, I. S., *Linear operators in spaces with indefinite metric*, John Wiley and Sons, New York, 1989.
- [4] Baidiuk, D.V., On the scattering matrices of the unitary extensions of isometric operators in the Pontryagin space, Trudy KROMSh, **19** (2009), 1–6.
- [5] Baidiuk, D.V., On boundary triplets and generalized resolvents of an isometric operators in a Pontryagin space, Ukrainian Mathematical Bulletin, vol 10, 2013, pp. 176–200.
- [6] Benewitz, C., Symmetric relations on a Hilbert space, Lect. Notes Math., **280**, 212–218 (1972).
- [7] Brodskii, M. Sm., Unitary operator colligations and their characteristic functions, Uspekhi Mat. Nauk, **33**, No. 4 (1978), 141–168.
- [8] Calkin, J.W. Abstract symmetric boudary conditions, Trans. Amer. Math. Soc., **45**, (1939), 369–442. (1939),
- [9] Derkach V. A. On indefinite abstract interpolation problem, Methods of Funct. Analysis and Topology, **7** (2001), no. 4, 87–100.
- [10] Derkach, V. A., Hassi, S., Malamud, M. M., and de Snoo, H. S. V., Boundary relations and their Weyl families, Transactions of AMS, Vol. 358, no. 12 (2006), 5351–5400.
- [11] Dijksma A., Langer H. and de Snoo H. S. V. Generalized coresolvents of standart isometric relations and generalized resolvent of standart symmetric relations in Kreĭn space, Operator theory: Advances and Applicatios, **48**, Birkhäuser Verlag, Basel, 1990. – p. 261–274.
- [12] Iokhvidov, I. S. and Kreĭn, M. G., The spectral theory of operators in spaces with indefinite metric, I, Trudy Mosk. Mat. Obshch., **5**, (1990) 367–432.
- [13] Iokhvidov, I. S., The description of the J -unitary extensions of J -isomeric operators with equal and finite defect numbers, DAN USSR, **228**, No. 4 (1976), 783–786.

- [14] Gorbachuk V.I. and Gorbahuk M.L. *Boundary value problems for operator-differential equations*, Naukova Dumka, Kiev, 1984, [Russian]
- [15] Katsnel'son, V. E., Kheifets, A. Ya., and Yuditskii, P.M., An abstract interpolation problem and the theory of the extensions of isometric operators, *Operators in Functional Spaces and Problems in Function Theory* (Russian), Naukove Dumka, Kiev, (1987), 83–96.
- [16] Langer G. K. The generalized coresolvents of a Π -isometric operator with unequal defect numbers. *Funkcional. Anal. i Prilozen.* 5 (1971), No. 4, 7375 (Russian).
- [17] Malamud, M. M., and Mogilevskii, V.I., The generalized resolvents of an isometric operator, *Mat. Zametki*, 73 No. 3 (2003), 460–466.
- [18] Malamud, M. M., and Mogilevskii, V.I., The resolvent matrices and the spectral functions of an isometric operator, *Docl. Akad. Nauk*, 395, No. 1 (2004), 11–17.
- [19] Nitz O. Generalized resolvents of isometric linear relations in Pontryagin space, 1: Foundations, *Operator theory: Advances and Applications*, vol. 118 (2000), 303–319.
- [20] Nitz O. Generalized resolvents of isometric linear relations in Pontryagin space, 2: Krein–Langer formula, *Methods of Functional Analysis and Topology*, vol. 6, No. 3 (2000), 72–96.
- [21] Shmul'yan, Yu. L., Theory of linear relations and spaces with indefinite metric, *Funkts. Anal. Pril.*, 10, No.1 (1976), 67–72.

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